SySc 512, Practice Final

1) a) $-\nabla V(x,y) = \begin{bmatrix} -(x+1) \\ -(2y) \end{bmatrix} \Rightarrow \begin{matrix} \dot{x} = -(x+1) = F_x \\ \dot{y} = -2y = F_y \end{matrix}$

b) $\dot{x} = 0 \Rightarrow x = -1$
$\dot{y} = 0 \Rightarrow y = 0$

c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, d) eigenvals $= -1, -2$

e) $\nabla V = 0 \Rightarrow (x,y) = (-1, 0)$

f) $H = \nabla^2 V = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ w/ e-vals $= -1, -2$

g) $-\nabla V(0,1) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

2) a) $\nabla f = \begin{bmatrix} x+y \\ x-1+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y$
but $x-1 = -y \Rightarrow$ ✗
No extremum!

b) $\mathcal{L} = f(x,y) - \lambda c(x,y)$
$= \frac{1}{2}x^2 + (x-1)y + \frac{1}{2}y^2 - \lambda(x-y-1)$

c) $\partial_x \mathcal{L} = x + y - \lambda = 0$ (1)
$\partial_y \mathcal{L} = x - 1 - \lambda = 0$ (2)
$c = x - y - 1 = 0$ (3)

eq (2) − eq (3) $\Rightarrow 2y = -\lambda$
eq (2) − eq (1) $\Rightarrow 2\lambda = 1$
$\Big\}\ 2y = -\frac{1}{2} \Rightarrow y = -\frac{1}{4}$

eq (3) $\Rightarrow x = y + 1 = 3/4 \Rightarrow (x^*, y^*, \lambda^*) = (\frac{3}{4}, -\frac{1}{4}, \frac{1}{2})$

$V(x^*, y^*) = \frac{1}{2}(\frac{9}{16}) + (-\frac{1}{4})(-\frac{1}{4}) + \frac{1}{2}(\frac{1}{16})$
$= \frac{1}{2}(\frac{9}{16} + \frac{2}{16} + \frac{1}{16}) = \frac{1}{2}(\frac{12}{16}) = \frac{3}{8}$

d) $\nabla^2 V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\nabla c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{n} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4 > 0$
$\Rightarrow$ min.

SySc 512 Practice Final

3) a) Complete the square to isolate x_2

Find some $a x_1^2 \ni \frac{x_1 x_2}{\sqrt{2}} = 2ab$ of $(a+b)^2$

Since $\left(x_2 - \frac{x_1}{2\sqrt{2}}\right)^2 = x_2^2 - x_1 x_2 \frac{1}{\sqrt{2}} + \frac{1}{4} x_1^2$

$\Rightarrow \frac{1}{2} x_1^2 - \frac{1}{8} x_1^2 + \frac{1}{8} x_1^2 - x_1 x_2 \frac{1}{\sqrt{2}} + x_2^2 =$

$= \frac{3}{8} x_1^2 + \left(x_2 - \frac{x_1}{2\sqrt{2}}\right)^2$

$\Rightarrow$

$$P(x_1, x_2) = \frac{\sqrt{2}}{4\pi} \exp\left[-\left(\frac{x_1^2}{2} + \frac{x_1 x_2}{\sqrt{2}} + x_2^2\right)\right]$$

$$= \frac{\sqrt{2}}{4\pi} \exp\left[-\left(\frac{3}{8} x_1^2 + \left(x_2 + \frac{x_1}{2\sqrt{2}}\right)^2\right)\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} P(x_1, x_2)\, dx_2 = \frac{\sqrt{2}}{4\pi} e^{-\frac{3}{8} x_1^2} \int_{-\infty}^{\infty} \exp\left[-\left(x_2 - \frac{x_1}{2\sqrt{2}}\right)^2\right] dx_2$$

$$= \frac{\sqrt{2}}{4\pi} e^{-\frac{3}{8} x_1^2} \sqrt{\pi} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{3}{8} x_1^2}$$

b) $\mathbb{E}\xi_1 = 0$ because $P_{\xi_1}(x_1)$ is centered about 0.

$\mathbb{D}\xi_1 = \mathbb{E}(\xi_1 - \mathbb{E}\xi_1)^2 = \mathbb{E}\xi_1^2$

$= \int_{-\infty}^{\infty} x_1^2 P_{\xi_1}(x_1)\, dx_1$

Since $\int_{-\infty}^{\infty} x^2 e^{-ax^2} = \frac{2 \cdot 3}{\sqrt{2^2 a^2}} \sqrt{\frac{\pi}{a}}$

$$\Rightarrow = \frac{3}{2} \frac{\sqrt{\pi}}{\sqrt{2\pi}} \frac{\sqrt{\frac{8}{3}}}{\sqrt{2^2 (3/8)^2}} = \frac{8\sqrt{\frac{8}{3}}}{2 \cdot 2} = 2\sqrt{\frac{8}{3}}$$

c) $P(B|A) = \frac{P(A,B)}{P(A)}$

$$\Rightarrow P(x_2|x_1) = \frac{P(x_1, x_2)}{P(x_1)} = \frac{\frac{\sqrt{2}}{4\pi} \exp\left[-\left(\frac{4}{8} x_1^2 - \frac{x_1 x_2}{\sqrt{2}} + x_2^2\right)\right]}{\frac{1}{2\sqrt{2\pi}} \exp\left[-\frac{3}{8} x_1^2\right]}$$

$$= \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{1}{8} x_1^2 - \frac{x_1 x_2}{\sqrt{2}} + x_2^2\right)\right]$$