SySc 512, Probability, Session (13)

Event A: one of N equaprobable, mutually exclusive, possibilities

Probability: $P(A) = \frac{N(A)}{N}$

w/ $N(A) \equiv$ # outcomes leading to A.

Examp 1 head on 2-sided coin:

$N(\text{head}) = 1, \quad N = 2$

$\Rightarrow P(A) = 1/2$

Find $P(A)$ by accumulating data

$n(A) =$ # occurances of A

$n =$ # trials

$\Rightarrow P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$

$\boxed{\text{Matlab Demo 1}}$ Coin flip (~~probability.m~~ coinFlip.m) $n: 10 \to 10k$

DeMere's paradox: 3 dice, seems like 11-spots are more common than 12-spots.

~~Were alternatives not equiprobable?~~

11 spots $(6,4,1)^6, (6,3,2)^6, (5,5,1)^3, (5\,4\,2)^6$
$(5\,3\,3)^3, (4,4,3)^3$

12 spots $(6,5,1)^6, (6,4,2)^6, (6,3,3)^3, (5,5,2)^3$
$(5,4,3)^6, (4,4,4)^1$

Same # of ways to get 11 or 12,

Pascal: But not equaprobable:

$N(11 \text{ spots}) = 3 \cdot 6 + 3 \cdot 3 = 27$

$N(12 \text{ spots}) = 3 \cdot 6 + 2 \cdot 3 + 1 = 25$

$\Rightarrow \boxed{\frac{27}{6^3} > \frac{25}{6^3}}$

$\boxed{\text{Matlab Demo 2}}$ (DeMere.m) $n: 10 \to 10k$

Combinatorics

Indep samples:

1) Given a_i w/ $i = 1, \ldots n_1$ & b_j w/ $j = 1, \ldots, n_2$
$\Rightarrow \exists\ n_1 n_2$ ordered pairs (a_i, b_j)

2) Given $a_i, b_j, \ldots x_k$, $k = 1, \ldots n_r$
$\Rightarrow \exists\ n_1 n_2 \cdots n_r$ r-tuples $(a_i, b_j, \ldots, x_k)$

SySc 512, Session 13, Prob (cont.)

Ordered Samples (sampling w/ replacement)

n-objects: $a_i \Rightarrow n^r$ ordered samples

$N(a_i, a_j, \ldots a_k) = n^r$

Sampling w/o replacement: Once placed, removed from population

$N = n(n-1)\cdots 2\cdot 1 \Rightarrow n!$ permutations

— 3) Subpopulations of size $r \le n$ elements:

$$C_r^n = \frac{n!}{r!(n-r)!} \equiv \binom{n}{r} \quad \begin{pmatrix} n \text{ things} \\ \text{taken} \\ r \text{ @ a time} \end{pmatrix} \text{ Binomial Coeffs.}$$

Proof:

If order is relevant

$\Rightarrow n$ ways of removing 1st

$(n-1)$ ways of removing 2nd

$(n-r+1)$ " " " rth

$\Rightarrow n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

If order irrelevant: $\frac{n!}{(n-r)!}$ groups of size r

But, overcounted by permutations of r

$\Rightarrow \frac{n!}{r!(n-r)!}$

(BTW: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$)

$\Rightarrow$ Pascals' triangle yields terms.

— 4) Partitions: n-elems,

Define $p_1, p_2, \ldots p_k \ni n = \sum_{i=1}^{k} p_i$

$$\Rightarrow N = \frac{n!}{p_1!\,p_2!\cdots p_k!} = \begin{cases} \text{ways to partition} \\ n\text{-elems into } k \text{ subpops.} \\ \text{of sizes } p_1, p_2, \ldots p_k \end{cases}$$

Proof: Successive use of $C_{p_i}^n$

$N = N_1 N_2 \cdots N_{k-1}$

$= C_{p_1}^{n} C_{p_2}^{n-p_1} \cdots C_{p_{k-1}}^{n-p_1-\cdots-p_{k-2}}$

[Rozanov, p 8]

SySc 512, Session 13, Prob (cont)

Independent Events

Sample point: ω (event)

Sample space: Ω (space of events)

Let $A = \{\omega_1, \ldots\}$ be a set of events

If 2 subsets of events: $A = A_1 \cup A_2$

$\Rightarrow A_1$ & A_2 are mutually exclusive

$$\Rightarrow \frac{n(A)}{n} = \frac{n(A_1)}{n} + \frac{n(A_2)}{n}$$

$$\text{or} \quad P(A) = P(A_1) + P(A_2)$$

Law of probabilities:

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

General Properties: for arbitrary events A_1 & A_2

1) $0 \le P(A) \le 1$, because $0 \le \frac{n(A)}{n} \le 1$

2) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

or, $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots$

Coincidences [Rozanov, p19]

n expr's, but n data files seperated from n lab notes (forgot numbers)

Question: What is the chance that 1 random file matches 1 random note?

Let A_k be event that k^{th} file $\leftrightarrow$ k^{th} note

$\Rightarrow$ Event that 1 file $\leftrightarrow$ its note is

$$A = \bigcup_{k=1}^{n} A_k \Rightarrow \text{calculate } P(A)$$

Each random match is permutation of files

$$N(A_{k_1} \cap \cdots \cap A_{k_m}) = (n-m)!$$

is permut. of $(n-m)$ things.

$N = n!$ (permut of n things)

And, $\exists$ C_m^n distinct events of type

$A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_m}$ (n taken m @ a time)

$$\Rightarrow \sum_{k_1 < \cdots < k_m} P(A_{k_1} \cap \cdots \cap A_{k_m}) = C_m^n \frac{(n-m)!}{n!} = \frac{1}{m!}$$

$$\Rightarrow P(A) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \pm \frac{1}{n!} \xrightarrow[n \to \infty]{} 1 - e^{-1} \approx 0.63$$

SySc 512, Session 13, Prob (cont.)

Dependent Events

Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

(assuming $P(B) > 0$) $= \frac{N(A \cap B)}{N(B)} = \left(\frac{\# \text{ both } A \& B}{\# B}\right)$

Properties:

1) $0 \leq P(A|B) \leq 1$

because $0 \leq P(A \cap B) \leq P(B)$, since $A \cap B \subset B$

2) If $A \cap B = \phi \Rightarrow P(A|B) = 0$

3) If $B \Rightarrow A$, $B \subset A \Rightarrow P(A|B) = 1$

4) If $A_1, \ldots$ are mutually exclusive, & $A = \bigcup_k A_k$

$\Rightarrow P(A|B) = \Sigma_k P(A_k|B)$

because $P(A \cap B) = \Sigma_k P(A_k \cap B)$

Suppose $\bigcup_k B_k = \Omega$ (exhaustive set)

and all B_k mutually exclusive.

$\Rightarrow A = \bigcup_k (A \cap B_k)$

$\Rightarrow P(A) = P(\bigcup_k (A \cap B_k))$

$= \Sigma_k P(A \cap B_k)$ (exclusive B_ks)

$= \sum_k P(A \cap B_k) \frac{P(B_k)}{P(B_k)} = \sum_k P(A|B_k) P(B_k)$

Bayes' Rule

Since, $P(A \cap B) = P(A|B) P(B)$

and, $= P(B|A) P(A)$

$\Rightarrow P(A|B) = P(B|A) \frac{P(A)}{P(B)}$

Bayesian Inference: H = hypothesis, E = evidence

$$P(H|E) = \frac{P(E|H)}{P(E)} P(H)$$

$P(H)$: "prior" probability that hypoth is true

$P(H|E)$: "posterior" probability that H is true given the evidence E.

$P(E)$: marginal prob of observing evidence E.

$P(E|H)$: "likelihood" of observing E if H is true

SySc 512, Session 13 (cont)

Statistical Independence

Many samples trials leading to both A_1 & A_2: $P(A_1 \cap A_2) \simeq \frac{n(A_1 \& A_2)}{n}$

and

$$P(A_2) \simeq \frac{n(A_2)}{n}$$

Looking only at trials w/ A_2, the occurrence of A_1 is

$$P(A_1) \simeq \frac{n(A_1 \& A_2)}{n(A_2)}$$

$$\Rightarrow P(A_1 \cap A_2) \simeq \frac{n(A_1 \& A_2)}{n(A_1)} \cdot \frac{n(A_2)}{n} \simeq P(A_1)P(A_2)$$

∴ statistically indep.

Condition for indep: $P(A_1 | A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = P(A_1)$

Mutually indep events:

$$P(A_i \cap A_j) = P(A_i)\,P(A_j)$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$$

$$\vdots$$

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$$

for all combos of indices.