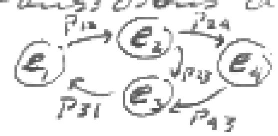
SySc 512, Session 15, Uncertain Dynamics

Markov Chains (Discrete Markov Process)

1) System w/ finite # states $\vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}$
2) Random state changes:
   Rand variable: $\xi(n)$: $\xi(0) \to \xi(1) \to \cdots$
3) Initial probability @ $n = 0$
   $$p_i^0 = P\{\xi(0) = e_i\}, \quad i = 1, \ldots$$
4) Markov condition: State transitions depend only on previous state




$$p_{ij} = P\{\xi(n+1) = e_j \mid \xi(n) = e_i\}$$
   w/ $i, j = 1, 2, \ldots$
   transition probabilities indep of $n$.

Compute: $p_j(n) = P\{\xi(n) = e_j\}$ (system is in state $e_j$ after $n$ steps)
After $n-1$ steps $\Rightarrow p_k(n-1)$
$$= \sum_k p_k(n-1) p_{kj}$$
$\Rightarrow$ recursive $\to p_j(0) = p_j^0$

To solve, choose a definite start state $i$:
$$p_j(0) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$
Then compute the conditional probability
$$p_{ij}(n) = p_j(n) = P\{\xi(n) = e_j \mid \xi(0) = e_i\}$$

The system goes from $e_i \to \cdots e_j$ in $n$-steps
$$= \sum_k p_{ik}(n-1) p_{kj}$$
$\Rightarrow$ matrix multiplication: Transition matrix
$$\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \\ \vdots & & \ddots \end{bmatrix} \Rightarrow \mathbb{P}(n) = \begin{bmatrix} p_{11}(n) & \cdots \\ \vdots & \end{bmatrix}$$
w/ $\mathbb{P}(0) = \mathbb{1}$, $\mathbb{P}(1) = \mathbb{P}, \ldots, \mathbb{P}(n) = \mathbb{P}^n$
$$\Rightarrow p_j(n) = [\mathbb{P}^n \vec{e}^0]_j$$

~~Matlab Demo / Markov~~

SySc 512 Session 15 Uncertain Dyns (cont'd)

Random Walk

$$p_{ij} = \begin{cases} p & \text{if } j=i+1 \\ 0 & \text{otherwise} \\ q & \text{if } j=i-1 \end{cases} \Rightarrow \mathbb{P} = \begin{bmatrix} 0 & p & 0 & \cdots \\ q & 0 & p & \\ 0 & q & 0 & \\ \vdots & & & \ddots \end{bmatrix}$$ n-states

Drift: $\mathbb{E}(\Delta\xi) = p-q$

because $\mathbb{P}\{\Delta\xi = +1\} = p$, $\mathbb{P}\{\Delta\xi = -1\} = q$

Diffusion:

$$\mathbb{D}(\Delta\xi) = \mathbb{E}(\Delta\xi)^2 - (\mathbb{E}\Delta\xi)^2$$
$$= 1-(p-q)^2$$
$$= (p+q)^2 - (p-q)^2 = 4pq$$

Examp: Start in center: $\vec{e}_0 = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow$ compute $\mathbb{P}^n \vec{e}_0$

Matlab \ MarkovWalk

Conform with RandomWalk

randomlines.swf

Ornstein-Uhlenbeck Process: Discrete

Confined walk: (N-steps) $\xi_n$ = state @ step-n

$$p_{ij} = \begin{cases} \frac{1}{2} - \frac{s_n - s_0}{2s_0} & \text{if } j = i+1 \\ 0 & \text{otherwise} \\ \frac{1}{2} + \frac{s_n - s_0}{2s_0} & \text{if } j = i-1 \end{cases}$$

Matlab \ UOwalk

SySc 512, Uncertain Dyns (cont.)
Continuous Markov Process $\Delta t \to 0$

1) Sys w/ finite # states $\bar{e} \in \mathbb{R}^N$
2) Evolution described by random var $\xi(t)$
3) Init probability @ $t=0$

$$p_i^0 = P\{\xi(0) = e_i\}$$

4) Given state $e_i$ at time $t'$ => transition prob

$$p_{ij}(t) = P\{\xi(t'+t) = e_j \mid \xi(t') = e_i\}$$

indep of $t'$ (~~[illegible]~~ history independent)

As for Markov chains

$$p_{ij}(t'+t) = \sum_k p_{ik}(t') p_{kj}(t)$$

However, continuous time transitions add a twist:
Let $\tau$ be (random) time to leave state $e$

$$\Rightarrow P\{\tau > t\} = \exp[-Rt] \quad , t \geq 0$$

w/ R is non-neg constant (transition rate)

Prf [Rozanov] Let $\varphi(t) = P\{\tau > t\}$
If $\tau > t'$ & Markov (history indep)

$$\Rightarrow \varphi = P\{\tau > t'+t \mid \tau > t'\} \quad (t' \text{ could be } 0)$$

$$P\{\tau > t'+t\} = P\{\tau > t'+t \mid \tau > t'\} P\{\tau > t'\}$$

$$\varphi(t+t') = \varphi(t)\varphi(t')$$

~~[illegible]~~ $\Rightarrow \ln \varphi(t+t') = \ln \varphi(t) + \ln(t')$

[Since, $f(x+y) = f(x) + f(y)$ implies f linear => proportionality: $f(x) = Ax$]

Then, let $\ln \varphi(t) = -Rt$

$$\therefore \quad P\{\tau > t\} = e^{-Rt}$$

Density function must be normalized;

$$A \int_0^\infty e^{-Rt} dt = 1$$

$$A\left[-\tfrac{1}{R} e^{-R(\infty)} - \left(-\tfrac{1}{R} e^{-R(0)}\right)\right] = A/R$$

$$\Rightarrow p_\tau(t) = \begin{cases} Re^{-Rt} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Poisson Distrib

SySc 512, Uncertain Dyns (cont.)

Stochastic Diff Eq w/ Restoring Force

$$\dot{x} = F(x) + \xi(t)$$

w/ $\xi(t)$ has distrib: $\mathbb{E}\,\xi(t) = 0$

$$\mathbb{E}(\xi(t)\xi(t')) = \sigma\,\delta(t-t')$$

Let $F = -\nabla V(x)$, $V(x) = \frac{a}{2}x^2$

$$\Rightarrow \dot{x} = -ax + \xi(t)$$

Matlab | OU_DEq_Euler

| OU_DEq_~

Stationary distrib: Normal

Gaussian Distrib

System Noise vs Measurement Noise

$$\dot{x} = F(x) + \xi_s(t) \quad \text{(sys noise)}$$

$$y = G(x) + \xi_M(t) \quad \text{(measure noise)}$$