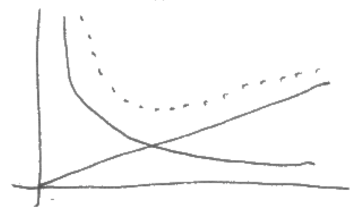
SySc 512, Session 7, Optimization

Housing Insulation examp:

x = thickness of insul

cx = cost per thickness

k/x = ~~fuel~~ cost



Total cost:

$$f(x) = k/x + cx$$

Minimize cost $\Rightarrow$ minimize $f(x)$

f is "objective function"

We seek a "minimizer", $x^* \in X$ (feasible set)

such that,

$$x^* = \min_{x \in X} f(x)$$

problem of opt. thy: ~~design~~

Conditions for local minimum at x^*:

1) 1^{st} order condition:

Necessary that: $\left.\frac{d}{dx} f(x)\right|_{x=x^*} = 0$

2) 2^{nd} order condition:

Necessary that: $\left.\frac{d^2}{dx^2} f(x)\right|_{x=x^*} \geq 0$

3) If (1), then

Sufficient that: $\left.\frac{d^2}{dx^2} f(x)\right|_{x=x^*} > 0$

Insulation example: ~~Also~~

1) $\frac{d}{dx} f(x) = 0 = -k/x^2 + c \Rightarrow x^2 = c/k$

$\Rightarrow x^* = \sqrt{c/k}$

2 $\frac{d^2}{dx^2} f(x) = 2\, k/x^3$

$\Rightarrow f''(x) > 0 \Rightarrow$ minimum

Examp 2: $f(x) = x^3$

$\Rightarrow f'(x) = 3x^2, \quad f''(x) = 6x$

1^{st} Cond) $f'(0) = 0 \Rightarrow x^* = 0$

2^{nd} Cond) $f''(x) = 0 \Rightarrow$ Not sufficient!

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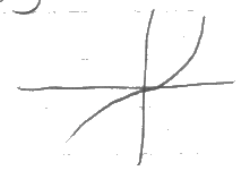
Examp 2) (cont) $\begin{cases} f''(x-\epsilon) < 0 \\ f''(x^* + \epsilon) > 0 \end{cases}$

However, since

$\Rightarrow$ not minimum



Examp 3)

$$f(x) = x^4 \Rightarrow f'(x) = 4x^3, \; f''(x) = 12x^2$$

1st Cond) $f'(0) = 0 \Rightarrow x^* = 0$

2nd Cond) $f''(0) = 0 \Rightarrow$ not sufficient

Again degenerate, but

$f''(x^* \pm \epsilon) > 0 \Rightarrow$ minimum.

Minima of Functions over $\bar{X} \in \mathbb{R}^n$

Use 1-dim trajectories to generalize conditions to n-dimensions.

Minimize the objective function $f(\vec{x}) : \bar{X} \subset \mathbb{R}^n \to \mathbb{R}$ (like a pot. fct.)

$\Rightarrow$ find $\vec{x}^*$ satisfying 1st & 2nd order conds.

Define trajectories through $\vec{x}^*$,

$$\vec{y}_s(t) = \vec{x}^* + t\vec{s}$$

$$\Rightarrow \frac{d}{dt}\vec{y}_s(t) = \vec{s}$$

Let $\vec{g}(\vec{x}) = \nabla f(\vec{x})$ (gradient of f)

Let $\mathbb{H}(\vec{x}) = \nabla[\nabla f]$ (Hessian of f)

$$H_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} f$$

Since, $\frac{d}{dt} f(\vec{y}_s(t))\Big|_{t=0} = \sum_k \frac{\partial}{\partial x_k} f(\vec{x}^*)\left(\frac{d}{dt}\vec{y}_s(0)\right)_k$

$$= \vec{s} \cdot \vec{g}(\vec{x}^*)$$

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1) 1st Order Conditions:

$\vec{s}\cdot\vec{g}(\vec{x}^*) = 0$ for any path through $\vec{x}^*$, (any $\vec{s}$)

$\Longleftrightarrow \boxed{\vec{g}(\vec{x}^*) = 0}$

2) 2nd Order Conditions:

$$\frac{d^2}{dt^2} f(\vec{\gamma}_s(t))\Big|_{t=0} = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} f(\vec{x})\Big|_{\vec{x}=\vec{x}^*} (\vec{\gamma}_s(0))_i \left(\tfrac{d}{dt}\vec{\gamma}_s(0)\right)_j$$

$$= \vec{s}^T \mathbb{H}(\vec{x}^*)\vec{s} \geq 0, \quad \forall \vec{s}$$

$\Longleftrightarrow \boxed{\mathbb{H}(\vec{x}^*) \text{ is non-negative definite}}$

$\left\{\begin{array}{l}\text{Non-neg def} \Longleftrightarrow \text{all eigenvalues non-negative} \\ \text{Positive def} \Longleftrightarrow \text{all " positive.}\end{array}\right.$

<u>Conditions for extremum</u>

<u>1st order</u> (Necessary) $\nabla f(\vec{x})|_{\vec{x}=\vec{x}^*} = 0$

<u>2nd order</u> (Sufficient)

If $\mathbb{H}(\vec{x}^*)$ positive definite $\Rightarrow$ min

If $\mathbb{H}(\vec{x}^*)$ negative definite $\Rightarrow$ max

Example 1) $f(x,y) = x^2 - y^2$

1st) $\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix} = 0 \ @ \ \vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2nd) $\mathbb{H}(\vec{x}) = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} f(\vec{x}) & \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(\vec{x}) \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(\vec{x}) & \frac{\partial}{\partial y}\frac{\partial}{\partial y} f(\vec{x}) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$\Rightarrow$ eigenvalues, $\lambda = \pm 2$

$\Rightarrow$ neither minimum nor max.

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Examp 2) $f(x,y) = x^2 + y^2 + axy$

1st order) $\nabla f = \begin{bmatrix} 2x + ay \\ 2y + ax \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ @ $\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2nd order) $\mathbb{H}(\vec{x}) = \begin{bmatrix} 2 & a \\ a & 2 \end{bmatrix}$ => eigenvalues?

Characteristic eq: $(2-\lambda)^2 - a^2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + (4-a^2) = 0$

$\Rightarrow \lambda = \dfrac{+4 \pm \sqrt{16 - 4(4-a^2)}}{2} = 2 \pm a$

=> Positive definite if $a < 2$

=> min.